STUDY OF SOLUTION OF SECOND ORDER DIFFERENTIAL EQUATIONS USING ORTHOGONAL COLLOCATION METHOD ON FINITE ELEMENTS

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Abstract

In this paper, the method of Orthogonal Collocation on finite elements (OCFE) is used to solve the second order differential equations. Two different examples are taken to study the behaviour of OCFE method to solve the differential equations. The solutions obtained from OCFE method for first example are compared with exact solution and solutions obtained from Orthogonal Collocation method (OCM). In case of second example, the numerical solutions obtained using OCFE method is compared with solutions obtained from Orthogonal Collocation method (OCM) and bvp4c MATLAB solver.

1. Introduction

Many problems in engineering and science can be formulated in terms of second order differential equations like vibration of spring, deflection of beams, simple harmonic motion, electric circuits etc. These equations can be solved by various analytical methods [1-2] but many of these equations cannot be solved exactly using analytic methods. To solve these type of differential equations, the various numerical methods have been developed over the years [2-5].

The second order ordinary differential equation is given by for $x \in (a, b)$

$$a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 f(u) = g(x) \tag{1}$$

with mixed boundary conditions $k_1 u + k_2 \frac{du}{dt} = l_1$ for $u = a$ and $k_3 u + k_4 \frac{du}{dt} = l_2$ for $u = b$ (2)

Weighted residual methods like Orthogonal Collocation method [6-9], Galerkin method, finite element methods [2-5] are very popular and useful methods to solve these ordinary as well as partial differential equations. Various authors has developed and extended the Orthogonal Collocation method on finite elements [10-13]. The OCFE method, in a way is generalization of OCM method. The method of orthogonal collocation on finite elements gives better results than orthogonal collocation method. The rate of convergence of OCFE method is better than OCM method.

2. Orthogonal Collocation on Finite Elements (OCFE)

In this method, the domain of x is divided into several sub domains, called finite elements. In each element, Orthogonal Collocation method is applied by taking n interior collocation points within that subdomain. The trial function is given by $\tilde{u} = \sum_{l=1}^{n+1} l_i(x) u(x_i)$ where $l_i(x) = \prod_{\substack{i=1 \\ i \neq j}}^{n+1} \frac{x - x_j}{x_i - x_j}$. $\frac{d\tilde{u}}{dx} = Au$ and $\frac{d^2\tilde{u}}{dx^2} = Bu$, where $A_{ij} = l_i'(x_j)$ and $B_{ij} = l_i''(x_j)$ Also As shown in Figure 1, domain $0 \le x \le 1$ is divided into N finite elements, with the i^{th} finite element being of length h_i and two interior collocation points within each finite element is taken. Thus the i^{th} finite element extends from $x = x_j = \sum_{l=1}^{i-1} h_l$ to $x = x_{j+3} = \sum_{l=1}^{i} h_l$, with $j = 3(i-1) + 1$. The collocation points are chosen to be zeros of Orthogonal polynomials like Jacobi polynomials, Shifted Legendre's polynomials, or Shifted Chebyshev's Polynomials in the interval [0, 1].

A variable, $\zeta^{<i>}$ is defined on the i^{th} element, such that, it is zero at its beginning and unity at its end. Therefore $\zeta^{<i>} = \dfrac{x - x_j}{h_i}$; $i = 1, 2, 3, \ldots, N$; $j = 3(i-1) + 1$. The location of two interior collocation points in the i^{th} element can easily be written in terms of $\zeta^{<i>}$ [2], as

$$\zeta_2^{<i>} = 0.2113 = \frac{x_{j+1} - x_j}{h_i}, \ \& \ \zeta_3^{<i>} = 0.7887 = \frac{x_{j+2} - x_j}{h_i}; \qquad i = 1,2,3 \ldots N;\ j = 3(i-1)+1 \tag{3}$$

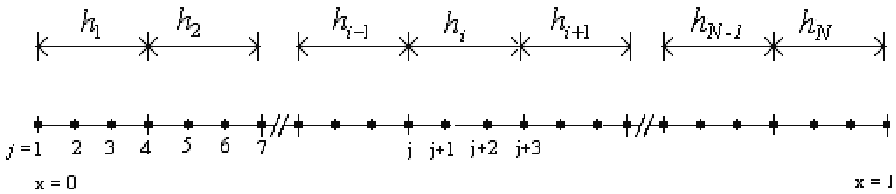

Figure 1: *Finite elements with two internal collocation points*

The orthogonal collocation is applied on element $\zeta^{<i>}$, as shown in Figure 2, the collocation equations in terms of the solutions at the collocation points are obtained. The function and its first derivative are assumed to be continuous at the node points of elements.

$$u^i\big|_{x_{i+}} = u^{i+1}\big|_{x_{i+1-}} \qquad \& \ \frac{du^i}{dx}\bigg|_{x_i+} = \frac{du^{i+1}}{dx}\bigg|_{x_{i+1}-} \tag{4}$$

At the 1st point of first element, the first boundary conditions is applied and at last point of last element, the second boundary condition is applied. After applying the method of OCFE a system of differential algebraic equations is obtained, which is solved to get the solution at the collocation points or grid points.

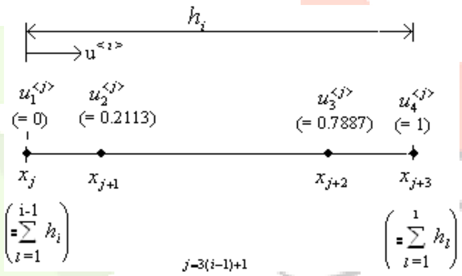

Figure 2: Details of the i^{th} finite element

3. Description of the Method

Consider the 2nd order differential equation with mixed boundary conditions

$$a_0 \frac{d^2u}{dx^2} + a_1 \frac{du}{dx} + a_2 f(u) = g(x) \qquad 0 \le x \le 1 \tag{5}$$

with boundary conditions

$$\mathrm{k}_1 \frac{du}{dx} + \mathrm{k}_2 u = l_1 \ \text{ at } \ x = 0 \tag{6}$$

$$\mathrm{k}_3 \frac{du}{dx} + \mathrm{k}_4 u = l_2 \ \text{ at } \ x = 1 \tag{7}$$

To understate the method, the problem is being presented for two finite elements of lengths h_1 and h_2,= 1- h_1 with each element having 2 interior collocation points.

Define $\zeta^{<l>}$ as $\zeta^{<l>} = \dfrac{x - x_l}{h_l}$ such that $0 \le \zeta^{<l>} \le 1$ within each element $[x_l, x_{l+1}]$. Then the eq. (5) can be written as:

$$\frac{a_0}{h_l^2}\frac{d^2u}{d\zeta^{<l>2}}+\frac{a_1}{h_l}\frac{du}{d\zeta^{<l>}}+a_2 f(u)=g(\zeta^{<l>}) \tag{8}$$

Now applying Orthogonal Collocation method on each element, using discritisation matrices A and B used in orthogonal collocation method, So for first element, eq.(5) can be written in the terms of $\zeta^{<1>}=\frac{x-x_1}{h_1}$ where $0\le\zeta^{<1>}\le 1$

$$\frac{a_0}{h_1^2}\frac{d^2u}{d\zeta^{<1>2}}+\frac{a_1}{h_1}\frac{du}{d\zeta^{<1>}}+a_2 f(u)=g(\zeta^{<1>}) \tag{9}$$

The discretized equations at points 2 and 3 can be written as:

At point 2: $$\frac{a_0}{h_1^2}\left[B_{21}u_1+B_{22}u_2+B_{23}u_3+B_{24}u_4\right]+\frac{a_1}{h_1}\left[A_{21}u_1+A_{22}u_2+A_{23}u_3+A_{24}u_4\right]+a_2 f(u_2)=g(\zeta_2^{<1>}) \tag{10}$$

At point 3: $$\frac{a_0}{h_1^2}\left[B_{31}u_1+B_{32}u_2+B_{33}u_3+B_{34}u_4\right]+\frac{a_1}{h_1}\left[A_{31}u_1+A_{32}u_2+A_{33}u_3+A_{34}u_4\right]+a_2 f(u_3)=g(\zeta_3^{<1>}) \tag{11}$$

Similarly, eq.(5) can be written in the terms of $\zeta^{<2>}=\frac{x-x_4}{h_2}$ where $0\le\zeta^{<2>}\le 1$ as follows:

$$\frac{a_0}{h_2^2}\frac{d^2u}{d\zeta^{<2>2}}+\frac{a_1}{h_2}\frac{du}{d\zeta^{<2>}}+a_2 f(u)=g(\zeta^{<2>}) \tag{12}$$

The discretized equations at points 5 and 6 can be written as:

At point 5: $$\frac{a_0}{h_1^2}\left[B_{21}u_4+B_{22}u_5+B_{23}u_6+B_{24}u_7\right]+\frac{a_1}{h_1}\left[A_{21}u_4+A_{22}u_5+A_{23}u_6+A_{24}u_7\right]+a_2 f(u_5)=g(\zeta_2^{<2>}) \tag{13}$$

At point 6: $$\frac{a_0}{h_1^2}\left[B_{31}u_4+B_{32}u_5+B_{33}u_6+B_{34}u_7\right]+\frac{a_1}{h_1}\left[A_{31}u_4+A_{32}u_5+A_{33}u_6+A_{34}u_7\right]+a_2 f(u_6)=g(\zeta_3^{<2>}) \tag{14}$$

Now, Point 4 is a common point of both the elements, *i.e.*, end point of first element and initial point of second element.

For element 1: $$\left|\frac{du}{dx}\right|_{x_4^-}=\frac{1}{h_1}\left|\frac{du}{d\zeta^{<1>}}\right|_{\zeta^{<1>}=1}=\frac{1}{h_1}\left[A_{41}u_1+A_{42}u_2+A_{43}u_3+A_{44}u_4\right] \tag{15}$$

For element 2: $$\left|\frac{du}{dx}\right|_{x_4^+}=\frac{1}{h_2}\left|\frac{du}{d\zeta^{<2>}}\right|_{\zeta^{<2>}=0}=\frac{1}{h_2}\left[A_{11}u_4+A_{12}u_5+A_{13}u_6+A_{14}u_7\right] \tag{16}$$

These slopes must be equal so that the polynomials in the two adjacent elements are smooth at the boundary point 4 therefore at point 4 one gets:

$$\frac{1}{h_1}\left[A_{41}u_1+A_{42}u_2+A_{43}u_3+A_{44}u_4\right]-\frac{1}{h_2}\left[A_{11}u_4+A_{12}u_5+A_{13}u_6+A_{14}u_7\right]=0 \tag{17}$$

Now solving two boundary conditions,

At point 1: The discretization equation is obtained by using boundary conditions for $x=0$ using $\left|\frac{du}{dx}\right|_{x=0}=\frac{1}{h_1}\left|\frac{du}{d\zeta^{<1>}}\right|_{\zeta^{<1>}=0}$

So, $$\mathrm{k}_1\frac{du}{dx}+\mathrm{k}_2 u=l_1 \quad\Rightarrow\frac{k_1}{h_1}\left[A_{11}u_1+A_{12}u_2+A_{13}u_3+A_{14}u_4\right]+k_2u_1=l_1 \tag{18}$$

At point 7: The discretization equation is obtained by using boundary conditions for $z=1$ using $\left|\frac{du}{dx}\right|_{x=1}=\frac{1}{h_2}\left|\frac{du}{d\zeta^{<2>}}\right|_{\zeta^{<2>}=1}$

$$k_3\frac{du}{dx}+k_4u=l_2 \qquad \Rightarrow \frac{k_3}{h_2}\left[A_{41}u_4+A_{42}u_5+A_{43}u_6+A_{44}u_7\right]+k_4u_7=l_2 \tag{19}$$

Collectively these equations can be put in a matrix form as follows

$$\begin{bmatrix} \frac{k_1A_{11}}{h_1}+k_2 & \frac{k_1A_{12}}{h_1} & \frac{k_1A_{13}}{h_1} & \frac{k_1A_{14}}{h_1} & 0 & 0 & 0 \\ \frac{a_0B_{21}}{h_1^2}+\frac{a_1A_{21}}{h_1} & \frac{a_0B_{22}}{h_1^2}+\frac{a_1A_{22}}{h_1} & \frac{a_0B_{23}}{h_1^2}+\frac{a_1A_{23}}{h_1} & \frac{a_0B_{24}}{h_1^2}+\frac{a_1A_{24}}{h_1} & 0 & 0 & 0 \\ \frac{a_0B_{31}}{h_1^2}+\frac{a_1A_{31}}{h_1} & \frac{a_0B_{32}}{h_1^2}+\frac{a_1A_{32}}{h_1} & \frac{a_0B_{33}}{h_1^2}+\frac{a_1A_{33}}{h_1} & \frac{a_0B_{34}}{h_1^2}+\frac{a_1A_{34}}{h_1} & 0 & 0 & 0 \\ \frac{A_{41}}{h_1} & \frac{A_{42}}{h_1} & \frac{A_{43}}{h_1} & \frac{A_{44}}{h_1}-\frac{A_{11}}{h_2} & -\frac{A_{12}}{h_2} & -\frac{A_{13}}{h_2} & -\frac{A_{41}}{h_2} \\ 0 & 0 & 0 & \frac{a_0B_{21}}{h_2^2}+\frac{a_1A_{21}}{h_2} & \frac{a_0B_{22}}{h_2^2}+\frac{a_1A_{22}}{h_2} & \frac{a_0B_{23}}{h_2^2}+\frac{a_1A_{23}}{h_2} & \frac{a_0B_{24}}{h_2^2}+\frac{a_1A_{24}}{h_2} \\ 0 & 0 & 0 & \frac{a_0B_{31}}{h_2^2}+\frac{a_1A_{31}}{h_2} & \frac{a_0B_{32}}{h_2^2}+\frac{a_1A_{32}}{h_2} & \frac{a_0B_{33}}{h_2^2}+\frac{a_1A_{33}}{h_2} & \frac{a_0B_{34}}{h_2^2}+\frac{a_1A_{34}}{h_2} \\ 0 & 0 & 0 & \frac{k_3A_{41}}{h_2} & \frac{k_3A_{42}}{h_2} & \frac{k_3A_{43}}{h_2} & \frac{k_3A_{44}}{h_2}+k_4 \end{bmatrix} \times \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \end{bmatrix} = \begin{bmatrix} l_1 \\ g(\zeta_2^{<1>}) \\ g(\zeta_3^{<1>}) \\ 0 \\ g(\zeta_2^{<2>}) \\ g(\zeta_3^{<2>}) \\ l_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -a_2f(u_2) \\ -a_2f(u_3) \\ 0 \\ -a_2f(u_5) \\ -a_2f(u_6) \\ 0 \end{bmatrix} \tag{20}$$

This system of equations can be written as:

$$\boldsymbol{MC=H+D} \tag{21}$$

This system of linear algebraic equations can be solved for solution vector by using any standard solution technique like Gauss elimination method, Gauss Jordon method, LU decomposition method. Software like C++, MATLAB and MATHEMATICA can be used for the solution of this system after proper programming.

4. Numerical Results

Example 1: Taking $a_0 = 1, a_1 = 4, a_2 = 4,\ g(u) = e^{-2x}$ and $f(u) = u$ in given equation (5). Also for boundary conditions $k_1 = 0, k_2 = 1, l_1 = 1\ and\ k_3 = 0, k_4 = 1, k_2 = 0.4737$

$$\frac{d^2u}{dx^2}+4\frac{du}{dx}+4u=e^{-2x} \qquad 0\le x\le 1 \tag{22}$$

with boundary conditions $u=1$ at $x=0$ & $u=0.4737$ at $x=1$. The exact solution of above stated problem is

$$u(x)=\left(1+2x+\frac{x^2}{2}\right)e^{-2x}$$

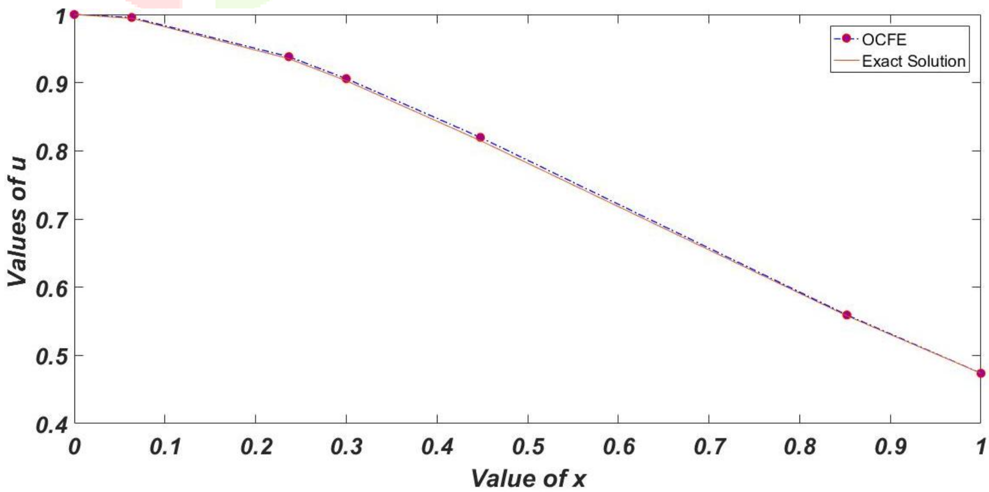

Figure 3: Exact Solution vs Numerical Solution using OCFE for Example 1

The numerical solution is obtained using Orthogonal Collocation with Finite Elements (OCFE) method by dividing the interval [0,1] in four uniform subdomains (i.e. n = 4) and Two collocations points within each element is taken. The collocation points are the roots of Shifted Legendre polynomial of second order in interval $[0, 1]$. Then results are compared with exact solution at these collocation points in Table 1. In Table 2, results obtained using OCFE are compared with Exact Solution and solution obtained using Orthogonal Collocation method (OCM).

*Table 1: Solution of Example **1** for n = 4 in each element using* $h_1 = 0.3$ ***and*** $h_2 = 0.7$

x	Exact Solution	Numerical Solution using OCFE	Absolute Error
0	1	1	0
0.052825	0.996052	0.996146	9.36903E-05
0.197175	0.953061	0.953107	4.66385E-05
0.25	0.92875	0.928722	2.77324E-05
0.302825	0.901257	0.901293	3.64496E-05
0.447175	0.815429	0.815457	2.78746E-05
0.5	0.781744	0.781733	1.08357E-05
0.552825	0.74754	0.747569	2.88479E-05
0.697175	0.654054	0.654085	3.13821E-05
0.75	0.620581	0.620592	1.13983E-05
0.802825	0.587805	0.587839	3.35291E-05
0.947175	0.502829	0.502866	3.68156E-05
1	0.4737	0.4737	0

Table 2: Comparison of OCM and OCFE results with Exact Solution for Example 1

x	Exact Solution	Numerical Solution using OCM	Numerical Solution using OCFE	Absolute Error_OCM	Absolute Error_OCFE
0	1	1	1	0	0
0.2113	0.946917	0.974456	0.946957	2.75383E-2	3.99448E-05
0.7887	0.596492	0.60208	0.596474	5.58784E-3	1.85589E-05
1	0.4737	0.4737	0.4737	0	0

Example 2: The differential equation used in isothermal Tubular reactor with axial mixing with an irreversible, second Order reaction taking place, described (in chemical engineering) by

$$\frac{1}{Pe}\frac{d^2c}{dz^2} - \frac{dc}{dz} - Da\ c^2 = 0 \qquad 0 \le z \le 1 \tag{23}$$

with boundary conditions $\frac{dc}{dz} = Pe(c-1)$ at $z = 0$ and $\frac{dc}{dz} = 0$ at $z = 1$, where, c is the dimensionless reactant concentration , z is the dimensionless axial position, *Pe* is Peclet number for mass transfer and *Da* is the Damkohler number. The numerical solution is obtained using Orthogonal Collocation with Finite Elements (OCFE) method by dividing the interval [0,1] in four uniform subdomains (i.e. n = 4) and Two collocations points within each element is taken. Table 3 and Table 4 shows solution for various values of Peclet number *Pe* and Damkohler number *Da*.

Table 3: *Concentration values 'c' at Collocation Points for Da = 5 at different Pe*

z	*Pe*=4	*Pe* =8	*Pe* =12	*Pe* =16	*Pe* =20	*Pe* =24	*Pe* =28	*Pe* =32
0	0.680283	0.765032	0.81068	0.840421	0.861679	0.877755	0.89039	0.900609
0.05283	0.617729	0.674964	0.703033	0.720286	0.732138	0.74085	0.747553	0.752885
0.19717	0.489082	0.504169	0.508197	0.509395	0.509607	0.509428	0.509088	0.508688
0.25	0.453375	0.460433	0.460741	0.459798	0.458631	0.457499	0.456468	0.455549
0.30283	0.421835	0.422684	0.420285	0.417854	0.415766	0.41402	0.412559	0.411328
0.44717	0.353088	0.343511	0.337183	0.332761	0.329508	0.32701	0.325027	0.323408
0.5	0.33301	0.321132	0.314171	0.309519	0.306188	0.303687	0.301739	0.300179
0.55283	0.315025	0.301215	0.293806	0.289019	0.285652	0.283157	0.28124	0.279725
0.69717	0.275552	0.257412	0.249228	0.244353	0.241015	0.238541	0.236614	0.235059
0.75	0.264374	0.244735	0.236326	0.23157	0.228449	0.226209	0.224505	0.223155
0.80283	0.254807	0.233409	0.224456	0.219574	0.216497	0.214373	0.212812	0.211614
0.94717	0.238622	0.212534	0.201014	0.194502	0.190308	0.187379	0.185215	0.183551
1	0.237223	0.210585	0.198704	0.191932	0.187541	0.184457	0.18217	0.180405

Table 4: *Concentration values 'c' at Collocation Points for Pe = 10 at different Da*

z	*Da*=2	*Da*=4	*Da*=6	*Da* =8	*Da* =10	*Da* =12	*Da* =15	*Da* =20

0	0.877729	0.813951	0.770828	0.738542	0.712939	0.691861	0.666088	0.633267
0.05283	0.817033	0.724387	0.662711	0.617012	0.581056	0.551636	0.515883	0.470698
0.19717	0.685021	0.551379	0.470504	0.414455	0.372572	0.339718	0.301467	0.255706
0.25	0.646321	0.506272	0.424511	0.369251	0.328753	0.297486	0.261672	0.219722
0.30283	0.611398	0.467135	0.38546	0.33137	0.292328	0.262547	0.228841	0.189944
0.44717	0.531884	0.384084	0.306068	0.256657	0.222129	0.196448	0.168089	0.136332
0.5	0.507544	0.360334	0.2843	0.236779	0.203881	0.179586	0.152938	0.123338
0.55283	0.485185	0.33909	0.265109	0.219421	0.188055	0.165038	0.139945	0.112269
0.69717	0.433145	0.291808	0.223461	0.182382	0.154705	0.134679	0.113133	0.089739
0.75	0.417212	0.277917	0.211509	0.171921	0.145396	0.126282	0.105796	0.083653
0.80283	0.402422	0.265241	0.2007	0.162515	0.137062	0.118789	0.099274	0.078267
0.94717	0.37338	0.24086	0.180144	0.14476	0.121412	0.104778	0.087136	0.068298
1	0.37053	0.238505	0.178176	0.14307	0.119929	0.103454	0.085993	0.067365

Table 5: *Concentration values 'c' at Collocation Points for Pe* = 16 *and Da=2*

z	OCFE	OCM	Solution using *bvp4c*	Absolute Error_OCM	Absolute Error_OCFE
0	0.912184	0.925619	0.911955081	1.3663763E-2	2.284230E-4
0.2113	0.684549	0.703926	0.684475416	1.9450926E-2	7.387000E-5
0.7887	0.40076	0.381949	0.400264475	1.8315342E-2	4.959580E-4
1	0.358754	0.358603	0.358751205	1.4816700E-4	2.59816E0-6

The numerical solutions are calculated using OCM method and OCFE method as well as using bvp4c MATLAB solver. Absolute Error of solutions of OCM and OCFE method are calculated from bvp4c solver.

5. Conclusion

As Table 2 for example 1 shows that the absolute errors in case of OCFE method is of order 10^{-5} as compared to absolute errors of order 10^{-2} in case of OCM. Also for example 2, it is observed that OCFE method gives the results more closer (of order 10^{-4}) to MATLAB *bvp4c* solver than results obtained from OCM method. The accuracy of OCFE can be increased by increasing number of subdomains i.e. elements. Also the number of collocation points within each element can be increased to obtain the better results. The results can be further studied by varying the choice of roots of Shifted Chebyshev Polynomial.

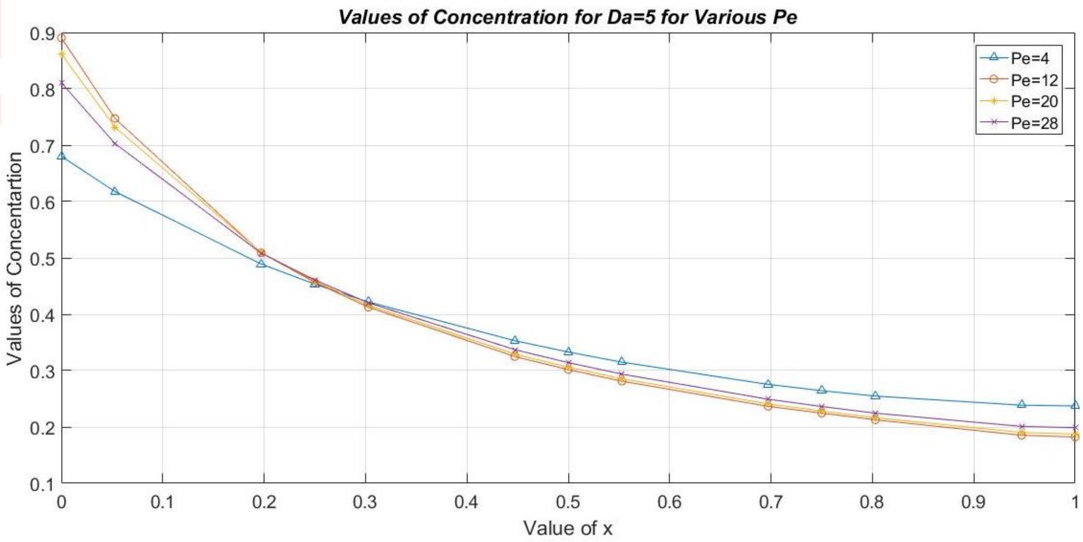

Figure 4: Concentration Values for Da= 5 for Different Peclet Number Pe

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